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The Crystallographic Groups and Weakly Amenable Groups

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Abstract

The purpose of this paper is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We prove that the Crystallographic groups need not be weak amenable.

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1 Introduction

From the point of view of noncommutative geometry, a C^*- is always considered as an algebra of continuous functions on some space. In this case of the reduced C^*- that space is a space of representations of the group. We detail provide a short survey of approximation properties of operator algebras associated with discrete groups. There are various notions of finite dimensional approximation properties for C^*- algebras and more generally operator algebras. Some of the notations(approximation properties) is defined in this paper, the reader is referred to [2], [12], [13], [6] and [19]: Haagerup discovery that is the reduced C^*- algebra, $\mathbf{F}_n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. Weak amenability is strictly weaker than amenability and passes to closed subgroups. It is proved by De Canni‘ere - Haagerup, Cowling and Cowling - Haagerup [5], [4] that real simple Lie groups of real rank one are weakly amenable (See also [18]), and by Haagerup [9] that real simple Lie groups of real rank at least two are not weakly amenable. For the latter fact, Haagerup proves that $SL(2,\mathbb{Z}) \rtimes \mathbb{Z}^2$ is not weakly amenable (See also [10]). The notion of weak amenability for groups was introduced by Cowling and Haagerup [4]. Author had prove that the Discrete Heisenberg group does not have the weakly amenable.

This paper is organized as follows. In section 2 we recall and prove some results about Approximation Property(AP), Weakly amenable and Herz- Schur multipliers.

Section 3 provides some detail of Crystallographic group and this group need not be weak amenability.

Our main result in this direction is the following.

Theorem 1.1. *The discrete Heisenberg Group need not be weak amenability.*

2 preliminaries

If we assume that G is a discrete group then the function δ_g form a basis for the Hilbert space $\ell^2(G)$ of square summable functions on G [6]. The group ring $\mathbb{C}[G]$ consists of all finitely supported complex-valued functions on G, that is of all finite combinations $f = \sum_{s\in G} a_s s$ with complex coefficients. Denote $B(\ell^2(G))$ the C^*- algebra of all bounded linear operator on Hilbert space $\ell^2(G)$.

Definition 2.1. [6] The *left regular representation*

$$\lambda : \mathbb{C}[G] \to B(\ell^2(G))$$

is defined by

$$\lambda(s)\delta_t(r) = \delta_t(s^{-1}r) = \delta_{st}(r) \text{ for } s, r \in G.$$

The reduced C^*- algebra $C^*_\lambda(G)$ of a group G arises from the study of the left regular representation λ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group.

Definition 2.2. [6] The *reduced group C^*- algebra G*, denoted by $C^*_r(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_\lambda = \|\lambda(c)\|$$

We give a general exposition of approximation properties which were initiated by Grothendieck [1]. His fundamental ideas have been applied to the study of groups and these noncommutative approximation properties have played a crucial role in the study of von Neumann algebras and C^*- algebra. Some weaker conditions (i.e., weak amenability and the approximation property) for locally compact groups have been studied by Haagerup and Kraus [8].

We begin with some definition of Haagerup and Kraus [8].

Definition 2.3. If A is a C^*- algebra, and $\mathcal{H}$ is a separable infinite Hilbert space, a net T_α in $CB(A)$ is said to converge in the *stable point-norm topology* to T in $CB(A)$ if $T_\alpha \otimes id_{K(H)}(a) \longrightarrow T \otimes id_{\mathcal{K}(\mathcal{H})}$ in norm for all $a \in A \otimes \mathcal{K}(H)$. Here $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators on H.

We recall the Fourier algebra

$$A(G) := \{f : f(t) = \langle \lambda(t)\xi \ , \ \eta\rangle \ \text{ for some } \ \xi, \eta \ \in L_2(G)\}$$

is the space of all coefficient function of the left regular representation λ. Given $f \ \in A(G)$, its norm is given by

$$\|f\| = \inf \{\|\xi\| \, \|\eta\| : \ f(t) = \langle \lambda(t)\xi \, , \, \eta\rangle\} \, .$$

With this norm, $A(G)$ is a Banach algebra with the point-wise multiplication [10].

Definition 2.4. A complex-valued function ϕ on G is a *multiplier* for $A(G)$ if the linear map $m_\phi(f) = \phi f$ sends $A(G)$ to $A(G)$.

A multiplier is a bounded and continuous function. Let $MA(G)$ denote the Banach space of multiplication of $A(G)$ equipped with the norm given by

$$\|\phi\|_{MA(G)} = \|m_\phi\|_{cb} \, ,$$

where $m_\phi : A(G) \longrightarrow A(G)$ denote the multiplier operator on $A(G)$ associated with ϕ.

Definition 2.5. A multiplier ϕ is called completely bounded if the operator $M_\phi : L(G) \longrightarrow L(G)$ induced by M_ϕ is completely bounded.

If the map m_ϕ is completely bounded on $A(G)$, we call ϕ a completely bounded multiplier of $A(G)$. The set of multipliers of $A(G)$ is denoted by $M_0A(G)$. If $\phi \in A(G)$ then ϕ is a bounded continuous function and M_ϕ is a bounded operator on the space $A(G)$. For $\phi \in A(G)$, let the map $m_\phi = m_\phi^*$ $m_\phi : A(G) \longrightarrow A(G)$ be defined by and $\overline{m_\phi}$ denote the restriction of m_ϕ to $C_r^*(G)$.

Definition 2.6. [10],[1] For a function ϕ on G and $C \geq 0$. We define the multiplier

$$m_\phi : \lambda(f) \longrightarrow \lambda(\phi f)$$

is completely bounded on $C^*_\lambda(G)$ and $\|m_\phi\|_{cb} \leq C$.

We let $M_oA(G)$ denote the space of all completely bounded multipliers of $A(G)$. Let $A(G) \subseteq M_oA(G)$, which is equipped with the cb-norm on $A(\Gamma)$. Therefore,

$$\|\phi\|_{M_oA(G)} = \|m\phi\|_{cb}.$$

If forms Banach space. It is known that

$$A(G) \subseteq M_oA(G) \subseteq MA(G)$$

If $\phi \in A(\Gamma) \subseteq M_oA(\Gamma)$, then the multiplication map is completely bounded by

$$\|M_\phi\|_{cb} \leq \|\phi\|.$$

Definition 2.7. [10] Let $\phi \in M_oA(G)$ if and only if there exist a Hilbert space $\mathcal{H}$ and bounded maps $p, q : G \longrightarrow \mathcal{H}$ such that

$$\phi(st^{-1}) = \langle p(s), q(s) \rangle \text{ for all } s, t \in G.$$

Here $\langle , \rangle$ denote the inner product on $\mathcal{H}$.

The completely bounded norm is given by

$$\|\phi\|_{M_oA(G)} = \inf \{\|p\|_\infty \|q\|_\infty\}.$$

Let $A_c(G)$ denote the space [10] of all elements in $A(G)$ with compact supports. We assume that $\phi_\alpha \in A_c(G)$, which means the ϕ_α have finite support.

Completely bounded Fourier multipliers naturally give rise to the formulation of a certain approximation property, namely weak amenability, which was studied extensively for Lie groups in [2], [6] and [19].

Definition 2.8. [2] The discrete group G is *amenable* if and only if there is a net (ϕ_α) in $A(G)$ with $\sup \|\phi_\alpha\|_{A(G)} < 1$, such that $\psi \in A(G)$. We have $\lim_\alpha \|\phi_\alpha \psi - \psi\|_{A(G)} = 0$.

Definition 2.9. [1] An approximate identity on G is a sequence (ϕ_n) of finitely supported functions such that ϕ_n uniformly converge to a constant function 1. We say that discrete G is *weakly amenable* if there is an approximate identity (ϕ_n) such that

$$C := \sup \|M_{\phi_n}\|_{cb} < \infty.$$

Definition 2.10. [8] The discrete group G has the *approximation property* (AP) if there is a net $\{\phi_\alpha\}$ in $A(G)$ such that $M_{\phi_\alpha} \longrightarrow id_{A(G)}$ in the stable point-norm topology on $A(G)$.

Definition 2.11. [1] A C^*- algebra A is *nuclear* if and only if it has the following *completely positive approximation property* (CPAP): The identity map on A can be approximated in the point norm topology by finite rank completely positive contractions.

Definition 2.12. [1] A C^*- algebra A has the metric approximation property (MAP) of Grothendieck if and only if the identity map on A can be approximated in the point-norm topology by a net of finite rank contractions.

Comparing the definitions we see that CPAP implies MAP (see for example [1]). Lance [17] has shown that G is amenable if and only if its reduced C^*- algebra A has the CPAP which is equivalent to $C_r^*(G)$ being nuclear. Completely positive maps are in particular completely bounded, which suggest the following weakening of the CPAP.

Definition 2.13. [1] A C^*-algebra A is said to have the *completely bounded approximation property* (CBAP) if there is a positive number C such that the identity map on A can be approximated in the point norm topology by a net $\{\phi_\alpha\}$ of finite rank completely bounded maps whose completely bounded norms are bounded by C.

The infimum of all values of C for which such constants exist is denoted by $\Lambda_{cb}(A)$ and is called the Cowling - Haagerup constant. We set $\Lambda_{cb}(G) = \infty$ if the locally compact group G does not have the CBAP. Obviously, a nuclear C^*- algebra has the metric approximation property. On the other hand, Haagerup [8] proved that the reduced C^*- algebra $\mathbb{F}_n$ has the metric approximation property, a very remarkable result since $C_r^*(\mathbb{F}_n)$, $n > 2$, is not nuclear, $\mathbb{F}_n$ not being amenable.

We have the following important result by Haagerup [8].

Theorem 2.14. *Let G be a discrete group. The following are equivalent:*

1. *G is weakly amenable,*
2. *$C_r^*(G)$ has the CBAP.*

Lemma 2.15. *An amenable discrete group is weakly amenable.*

Amenability of a group G implies weak amenability with $\Lambda(G) = 1$. Weak amenability was first studied in [8], in which de Canni'ere and the Haagerup [5] proved that the free group $\mathbf{F}_n$ on n generators with $n \geq 2$ is weakly amenable with $\Lambda(F_n) = 1$. This also implied that weak amenability is strictly weaker than amenability, since F_n is not amenable. The constant $\Lambda(G)$ is known for every connected simple Lie group G and depends on the real rank of G. First, note that if G has real rank zero, then G is amenable. A connected simple Lie group G with real rank one is locally isomorphic to one of the groups $SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$, with $n \geq 2$, or to $F4(-20)$.

Haagerup proved that all connected simple Lie groups with finite center and real rank greater than or equal to two are not weakly amenable [9]. Later, Dorofaeff proved that this result also holds for such Lie groups with infinite center [7].

A weaker approximation property defined in terms of completely bounded Fourier multipliers was introduced by the Haagerup and Kraus [8]

A detailed characterisation of AP is provided in [8]. Roe [19] considered the discrete group of the reduced group C^*- algebra of $C^*_r(G)$ is the fixed point algebra $\{Ad\rho(t) : t \in G\}$ acting on the uniform Roe algebra $C^*_u(G)$ [19]. A discrete group G has natural coarse structure which allows us to define the uniform Roe algebra, $C^*_u(G)$ [19]. We say that the uniform Roe algebra, $C^*_u(G)$, is the C^*- algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. The reduced C^*- algebra $C^*_r(G)$ is naturally contained in $C^*_u(G)$ [19]. According to Roe [19] G has the invariant approximation property (IAP) if

$$C^*_\lambda(G) = C^*_u(G)^G.$$

Next, we define the set of fixed points of $C^*_u(G, S)^G$ [15]:

Definition 2.16.

$$C^*_u(G, S)^G = \{T \in C^*_u(G, S)\, ;\, Ad(\rho_t \otimes id)T = T \text{ for all } t \in G\}\, .$$

We define Joachim Zacharias's IAP with coefficients (SIAP):

Definition 2.17. [20] We say that a discrete group G has the *strong invariant translation approximation property* (SIAP) if for any closed subspace S of the compact operators $\mathcal{K}$ (on $\ell^2(\mathbb{N})$). We have an isomorphism

$$C^*_u(G, S)^G = C^*_\lambda(G) \otimes S \text{ holds.}$$

Author also prove that the stability properties of the strong invariant approximation property [13]and Strong invariant approximation property for discrete groups [12].

Proposition 2.18. *[8] The semi direct product of two discrete groups with the AP has the AP.*

Remark 2.19. For discrete groups we have the following implications:

$$\text{Amenability} \implies \text{ weak amenability} \Longrightarrow \text{AP} \Longrightarrow \text{ exactness.}$$

3 Crystallographic Groups

The symmetry group of a tiling pattern of the plane is called a *crystallographic group*. In two dimensions there are 17 such groups which are also called *wallpaper groups* or *plane groups*. In three dimensions there are 230 crystallographic space group types [6]. The seventeen groups (*wallpaper groups* or *plane groups*) [6] are all extensions of an abelian group of translations isomorphic to $\mathbb{Z}^2$ by a finite group. A crystallographic group itself describes internal symmetries of a crystal.

Definition 3.1. The affine isometrics of the *Euclidean space* $\mathbb{E}^d$ are functions $f : \mathbb{E}^d \longrightarrow \mathbb{E}^d$, defined $f(x) = Ax + b$, for any $x \in \mathbb{E}^d$, where $A \in O(d)$, and $b \in \mathbb{E}^d$.

Definition 3.2. A *Crystallographic* group is a discrete group of isometrics of $\mathbb{E}^3$:

$$G = \{f = [x \longrightarrow Ax + b]; A \in O(d) \text{ and } b \in G\},$$

where $G \subset O(3)$ is a finite group and G contains an abelian free group generated by three linearly independent translations of $\mathbb{E}^3$.

There are exactly 32 types of finite subgroups $G \subset O(3)$ corresponding to all crystallographic groups [6].

Definition 3.3. A $d-$dimensional crystallographic group G is a discrete co-compact group of isometrics of $d-$dimensional Euclidean space $\mathbb{E}^d$.

Definition 3.4. Two *Crystallographic* groups G and G'' are said to be of the same type if there exists $T \in SO(3)$ such that $G'' = TGT^{-1}$.

Let g be any element of G. Every isometry of $\mathbb{E}^d$ can be written as a pair (M, v). Thus $g = (M, v)$, where M is a d-dimensional orthogonal matrix and v is a d-dimensional vector. The action of the pair (M, v) on $\mathbb{E}^d$ is defined by :

$$(M, v)w := Mv + w$$

$$(M, v).(N, w) := (MN, Mw + v).$$

An isometry (M, v) is a pure translation if M is the identity matrix. The set of translations of a given isometry group G forms a normal subgroup $T(G)$, which is the kernel of the homomorphism [11]:

$$\rho : G \longrightarrow O(d) \text{ by } (M, v) \longmapsto M.$$

In other words there is an exact sequence:

$$1 \longrightarrow T(G) \xrightarrow{i} G \xrightarrow{\rho} \rho(G) \longrightarrow 1,$$

where $\mathbb{D} := T(G)$ is a finitely generated abelian group and $\rho(G)$ is called the point group of G. Let G be a discrete group with an abelian normal subgroup $A := \rho(G)$ such that $\mathbb{D} := T(G) = G/A$ is a finite group of order n.

Let the quotient map be denoted by π. Let a cross-section $\gamma : T(G) \longrightarrow G$. Thus $\pi\gamma = id$ and $\gamma(e_{\mathbb{D}}) = e$, where e is the identity element of G and $e_{\mathbb{D}}$ is the identity element of $T(G)$.

Lemma 3.5. *[14] Let $\phi : G \longrightarrow \mathbb{D} \times A$, where $\mathbb{D} = G/A$, be defined as follows: $\phi(g) = (gA, \rho(g))$. Then the map ϕ is an isomorphism.*

Lemma 3.6. *[14] Let the quotient group $\mathbb{D}$ act on A by conjugation. Take $\theta : \mathbb{D} \longrightarrow Aut(A)$ defined as follows:*

$$\theta_d(a) = \gamma(d)a\gamma(d)^{-1} \; for \; d \in \mathbb{D}, a \in A.$$

Then θ is a homomorphism of $\mathbb{D}$ into $Aut(A)$

Theorem 3.7. *The Crystallographic groups which is not weakly amenable.*

Proof: If $T(G)$ spans $\mathbb{R}^d$, then $T(G)$ is a maximal abelian subgroup of G. Let $a = (I, v) \in T(G)$ and $b = (M, w) \in G$. We assume that b commutes with every element in $T(G)$, we have

$$(I, v)(M, w) = (M, w)(I, v)$$

$$(IM, Iw + v) = (MI, Mv + w)$$

so, $Mv = v$. It follows that $M \equiv I$ and b is a translation. The translation subgroup of a crystallographic group is discrete and is isomorphic to $\mathbb{Z}^m$ for some $m \leq d$. We have mentioned that $\mathbb{H}_3$ can be viewed as the semi direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. We have $\mathbb{H}_3 = \mathbb{Z}^2 \rtimes \mathbb{Z}$. And so there is an exact sequence:

$$1 \longrightarrow T(G) \xrightarrow{i} G \xrightarrow{\rho} \mathbb{Z}^m \longrightarrow 1,$$

Since $T(G)$ and $\mathbb{Z}^d$ are finitely generated and they have AP and also weakly amenable. In this way, all crystallographic groups are extensions of abelian groups of translations isomorphic to $\mathbb{Z}^m$ by a finite group [11]. Since semi direct product of weakly amenable group have not weakly amenable group. Therefore crystallographic groups is not weakly amenable.

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